On Type-2 Soft Groups

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Abstract

In the present paper, a notion of type-2 soft group is introduced and some of its important properties are studied. The behaviour of homomorphic image and pre-image of type-2 soft groups under type-2 soft mappings are discussed. Fundamental homomorphism theorems established in type-2 soft groups.

Keywords: Soft sets, Type-2 soft sets, Soft groups, Type-2 soft groups, Soft mappings, Type-2 soft mappings.

1. Introduction

In dealing with the uncertainties which is the most prevalent aspect in the natural occurrence of events, various theories such as probability theory, fuzzy set theory, rough set theory, soft set theory etc. were developed. In fuzzy set theory (Zadeh, 1965) and type-2 fuzzy set theory (Zadeh, 1975), (Mendal, 2009) crucial role is played by the membership function depending on various factors and the major complexities encountered while dealing with the theory of fuzzy sets and type-2 fuzzy sets. In 1999, Molodtsov tried to approach in an alternative way. Instead of taking the membership function he took a parameterized family of sets and called it soft set. He pointed out that Zadeh's fuzzy sets were special types of soft sets. In this respect it might be stated that the parameterization technique of soft set theory is more user friendly as compared to the membership function approach of fuzzy set theory in the field of their applications in real life problems. Afterwards Maji et al. (2003-05) work on some mathematical aspects of soft sets. Ever since mathematical activities are going on with all these concepts individually as well as their hybridizations such as soft group (Aktas & Cagman, 2007), fuzzy soft groups (Nazmul & Samanta, 2011), intuitionistic L-fuzzy soft groups (Nazmul & Samanta, 2013), soft topology (Shabir &Naz, 2011), (Majumdar & Samanta 2010), (Aygunolu & Aygun, 2012), soft topological group (Nazmul & Samanta, 2012-15) etc. In 2015, R. Chatterjee et al. (2015) proposed a parameterized structure for type-2 fuzzy sets and named it type-2 soft sets. As a continuation and observing a huge potential of soft set theory, it is natural to investigate the behaviour of group structures in type-2 soft set settings. The author already studies the properties of soft group and fuzzy soft group under soft mappings (Nazmul, 2017) and the behavior of type-2 soft mappings (Nazmul, 2017). In this paper, we have introduced a notion of type-2 soft groups and investigate some of its important properties. Homomorphism image and pre-image of type-2 soft groups under type-2 soft mappings introduced are also studied. Celebrated fundamental homomorphism theorems are established in type-2 settings.

2. Preliminaries

In this section following (Molodtsov, 1999), (Maji et. al., 2003), (Aktas & Cagman, 2007), (Nazmul & Samanta, 2014), (Chatterjee et. al., 2015), (Nazmul, 2017), some definitions and results of soft sets, soft groups, type-2 soft sets, and type-2 soft mappings are given in our form. Unless otherwise stated,
X will be assumed to be an initial universal set, A will be taken to be a set of parameters, \( P(X) \) denote the power set of X.

**Definition 2.1.** Let X be an initial universal set, A be the set of parameters, \( P(X) \) denotes the power set of X. A pair \((F,A)\) where \( F \) is a mapping from \( A \) to \( P(X) \), is called a soft set or type-I soft set over X. Also \( S_1(X,A) \) denotes the set of all soft set or type-I soft set over X under the parameter A.

**Definition 2.2.** Let \( \{(F_i,A);i \in I\} \) be a nonempty family of soft sets in \( S_1(X,A) \). Then their
(i) Intersection, denoted by \( \cap_{i \in I} \), is defined by \( \cap_{i \in I}(F_i,A) = (\cap_{i \in I} F_i,A), \forall \alpha \in A. \)
(ii) Union, denoted by \( \cup_{i \in I} \), is defined by \( \cup_{i \in I}(F_i,A) = (\cup_{i \in I} F_i,A), \forall \alpha \in A. \)
(iii) AND, denoted by \( \wedge_{i \in I} \), is defined by \( \wedge_{i \in I}(F_i,A) = (\wedge_{i \in I} F_i,A'), \forall (\alpha_i) \in A'. \)
(iv) OR, denoted by \( \vee_{i \in I} \), is defined by \( \vee_{i \in I}(F_i,A) = (\vee_{i \in I} F_i,A'), \forall (\alpha_i) \in A. \)

**Definition 2.3.** (Kharal, 2010; Majumdar et. al., 2010) Let \( S_1(X,A) \) and \( S_1(Y,B) \) be the families of all soft sets over X and Y respectively. The mapping \( f : S_1(X,A) \rightarrow S_1(Y,B) \) is called a soft mapping from X to Y, where \( f : X \rightarrow Y \) and \( \varphi : A \rightarrow B \) are two mappings. Also
(i) the image of a soft set \((F,A) \in S_1(X,A)\) under the mapping \( f_\varphi \) is denoted by \( f_\varphi((F,A)) = (f_\varphi(F),B) \), and is defined by, \( \forall \beta \in B, \)
\[
[f_\varphi(F)](\beta) = \begin{cases} \bigcup_{\alpha \in f_\varphi^{-1}(\beta)} [f(F(\alpha))] & \text{if } \varphi^{-1}(\beta) \neq \phi \\ \phi & \text{otherwise} \end{cases}
\]
(ii) the inverse image of a soft set \((G,B) \in S_1(Y,B)\) under the mapping \( f_\varphi \) is denoted by \( f_\varphi^{-1}((G,B)) = (f_\varphi^{-1}(G),A) \), and is defined by \( [f_\varphi^{-1}(G)](\alpha) = f_\varphi^{-1}([G(\varphi(\alpha))]), \forall \alpha \in A. \)
(iii) the soft mapping \( f_\varphi \) is called injective (surjective) if \( f \) and \( \varphi \) are both injective (surjective).
(iv) the soft mapping \( f_\varphi \) is identity soft mapping, if \( f \) and \( \varphi \) are both classical identity mappings.

**Definition 2.4.** (Chatterjee et. al., 2015) The pair \([\mathcal{F}, A]\) where \( \mathcal{F} \) is a mapping from \( A \) to \( S_1(X,A) \) is called type-2 soft set over \((X,A)\).
In this case, corresponding to each parameter \( \alpha \in A, \mathcal{F}(\alpha) \) is a type-1 soft set over \((X,A)\). Thus for each \( \alpha \in A, \exists \) a type-1 soft set \((F_\alpha,A)\) such that \( \mathcal{F}(\alpha) = (F_\alpha,A) \) where \( F_\alpha : A \rightarrow P(X) \). Also, \( S_2(X,A) \) denotes the set of all type-2 soft sets over \((X,A)\).

**Definition 2.5.** (Chatterjee et. al., 2015) Let \([\mathcal{F}, A], [\mathcal{G}, A] \in S_2(X,A)\). Then
(i) \([\mathcal{F}, A] \) is said to be soft subset of \([\mathcal{G}, A] \) if \( \mathcal{F}(\alpha) \subseteq \mathcal{G}(\alpha), \forall \alpha \in A.\) This relation is denoted by \([\mathcal{F}, A] \subseteq [\mathcal{G}, B] \).
(ii) \$ [\mathcal{F}, A] \$ is said to be soft equal to \([\mathcal{G}, A] \) if \([\mathcal{F}, A] \subseteq [\mathcal{G}, A] \) and \([\mathcal{G}, A] \subseteq [\mathcal{F}, A] \).
(iii) the complement of a type-2 soft set \([\mathcal{F}, A]\) is defined as \([\mathcal{F}, A]^c = [\mathcal{F}^c, A]\), where \(\mathcal{F}^c (\alpha) = [\mathcal{F}(\alpha)]^c\), for all \(\alpha \in A\).

(iv) \([\mathcal{F}, A]\) is said to be a type-2 null soft set if \(\mathcal{F}(\alpha) = (\emptyset, A)\), for all \(\alpha \in A\). This is denoted by \([\emptyset, A]\).

(v) \([\mathcal{F}, A]\) is said to be a type-2 absolute soft set if \(\mathcal{F}(\alpha) = (X, A)\), for all \(\alpha \in A\). This is denoted by \([X, A]\).

**Definition 2.6.** (Chatterjee et al., 2015) Let \([\mathcal{F}, A], [G, A] \in S_2 (X, A)\). Then their

(i) **Union**, is a type-2 soft set \([\mathcal{H}, A]\), denoted by \([\mathcal{F}, A] \cup [G, A] = [\mathcal{H}, A]\), is defined by \(\forall \alpha \in A, \mathcal{H}(\alpha) = \mathcal{F}(\alpha) \cup \mathcal{G}(\alpha)\).

(ii) **Intersection**, is a type-2 soft set \([\mathcal{H}, A]\), denoted by \([\mathcal{F}, A] \cap [G, A] = [\mathcal{H}, A]\), is defined by \(\mathcal{H}(\alpha) = \mathcal{F}(\alpha) \cap \mathcal{G}(\alpha), \forall \alpha \in A\).

(iii) **AND**, is a type-2 soft set \([\mathcal{H}, A \times A]\), denoted by \([\mathcal{F}, A] \land [G, A] = [\mathcal{H}, A \times A]\), is defined by \(H(\alpha, \beta) = \mathcal{F}(\alpha) \land \mathcal{G}(\beta), \forall (\alpha, \beta) \in (A \times A)\).

(iv) **OR**, is a type-2 soft set \([\mathcal{H}, A \times A]\), denoted by \([\mathcal{F}, A] \lor [G, A] = [\mathcal{H}, A \times A]\), is defined by \(H(\alpha, \beta) = \mathcal{F}(\alpha) \lor \mathcal{G}(\beta), \forall (\alpha, \beta) \in (A \times A)\).

**Definition 2.7.** (Nazmul, 2017) Let \(S_2(X, A)\) and \(S_2(Y, B)\) be the families of all type-2 soft sets over \((X, A)\) and \((Y, B)\) respectively. The mapping \(f : S_2(X, A) \rightarrow S_2(Y, B)\) is called a type-2 soft mapping, where \(f : S_1(X, A) \rightarrow S_1(Y, B)\) is a type-1 soft mapping and \(\psi : A \rightarrow B\) is a mapping. Also

(i) the image of a type-2 soft set \([\mathcal{F}, A] \in S_2(X, A)\) under the mapping \(f\), is denoted by \(f_\psi ([\mathcal{F}, A]) = [f_\psi (\mathcal{F}), B]\), and is defined by, \(\forall \beta \in B\),

\[
[f_\psi (\mathcal{F})](\beta) = \begin{cases} 
\bigcup_{\alpha \in \mathcal{F}(\alpha) \neq \emptyset} f_\psi (\mathcal{F}(\alpha)) & \text{if } \psi^{-1}(\beta) \neq \emptyset \\
(\tilde{\emptyset}, B) & \text{otherwise}
\end{cases}
\]

(ii) the inverse image of a type-2 soft set \([\mathcal{G}, B] \in S_2(Y, B)\) under the mapping \(f\), is denoted by \(f_\psi^{-1} ([\mathcal{G}, B]) = [f_\psi^{-1} (\mathcal{G}), A]\), and is defined by \(f_\psi^{-1} (\mathcal{G})(\alpha) = f_\psi^{-1} [\mathcal{G}(\psi(\alpha))], \forall \alpha \in A\).

(iii) The soft mapping \(f_\psi\) is called injective (surjective) if \(f\) and \(\psi\) are both injective (surjective).

(iv) The soft mapping \(f_\psi\) is identity soft mapping, if \(f\) and \(\psi\) are both identity mappings.

**Proposition 2.8.** (Nazmul, 2017) Let \(X\) and \(Y\) be two nonempty sets and \(f_\psi : S_2(X, A) \rightarrow S_2(Y, B)\) be a type-2 soft mapping. If \([\mathcal{F}, A], [\mathcal{F}_i, A] \in S_2(X, A)\) and \([\mathcal{G}, B], [\mathcal{G}_i, B] \in S_2(Y, B), i \in \Delta\). Then

(i) \([\mathcal{F}_1, A] \subseteq [\mathcal{F}_2, A] \Rightarrow f_\psi ([\mathcal{F}_1, A]) \subseteq f_\psi ([\mathcal{F}_2, A]);

(ii) \([\mathcal{G}_1, B] \subseteq [\mathcal{G}_2, B] \Rightarrow f_\psi^{-1} ([\mathcal{G}_1, B]) \subseteq f_\psi^{-1} ([\mathcal{G}_2, B]);

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In this section, $X, Y$ are taken to be groups, $e_x, e_y$ are the identity elements of $X, Y$ respectively.

**Definition 2.9.** (Aktas & Cagman, 2007; Nazmul & Samanta, 2011) Let $(F, A)$ be a type-1 soft set over a group $X$. Then $(F, A)$ is said to be a type-1 soft group or simply soft group over $X$ iff $F(\alpha)$ is a subgroup of $X, \forall \alpha \in A$ i.e. $F(\alpha) \leq X, \forall \alpha \in A$

**Proposition 2.10.** (Aktas & Cagman, 2007; Nazmul & Samanta, 2011) Let $(F_1, A)$ and $(F_2, A)$ be two type-1 soft groups over $X$. Then $(F_1, A) \cap (F_2, A)$ and $(F_1, A) \cap (F_2, A)$ are soft groups over $X$.

**Definition 2.11.** (Aktas & Cagman, 2007; Nazmul & Samanta, 2011) Let $(F, A), (F_1, A)$ and $(F_2, A)$ be soft groups over $(X, A)$. Then

(i) $(F, A)$ is said to be an identity soft group (absolute soft group) over $X$ if $F(\alpha) = \{e_X\} \left(F(\alpha) = X\right), \forall \alpha \in A$.

(ii) $(F_1, A)$ is said to be a soft subgroup (soft normal subgroup) of $(F_2, A)$, denoted by $(F_1, A) \triangleleft (F_2, A)$ if $F_1(\alpha) \leq F_2(\alpha) \left(F_1(\alpha) < F_2(\alpha)\right), \forall \alpha \in A$.

**Definition 2.12.** (Aktas & Cagman, 2007; Nazmul & Samanta, 2011) A soft function $f : S(X, A) \rightarrow S(Y, B)$ is said to be soft homomorphism (soft isomorphism) if $f : X \rightarrow Y$ is an algebraic homomorphism (isomorphism).

**Proposition 2.13.** (Nazmul, 2017) Let $(F, A), (F_1, A), (F_2, A)$ be soft groups over $(X, A)$ and $(G, B), (G_1, B), (G_2, B)$ be soft groups over $(Y, B)$. Also let $f : S(X, A) \rightarrow S(Y, B)$ be a soft mapping.

(i) If $(F, A)$ be identity soft group over $(X, A)$ and $f$ is a soft homomorphism such that $\varphi$ is onto, then $f(\varphi)(F, A)$ is identity soft group over $(Y, B)$.

(ii) If $(G, B)$ be identity soft group over $(Y, B)$ and $f$ is a soft homomorphism, then $f^{-1}(G, B)$ is the constant soft group $(ker(f), A)$ over $(X, A)$. In particular, if $f$ is one-one then $f^{-1}(G, B)$ is identity soft group over $(X, A)$.
(iii) If \((F, A)\) be absolute soft group over \((X, A)\) and \(f_\varphi\) is a soft homomorphism such that \(\varphi\) is onto, then \(f_\varphi[(F, A)]\) is a constant soft group over \((Y, B)\). Also if \(f_\varphi\) is onto, then \(f_\varphi^{-1}[(F, A)]\) is absolute soft group over \((Y, B)\).

(iv) If \((G, B)\) be absolute soft group over \((Y, B)\) and \(f_\varphi\) is a soft homomorphism, then 
\[f_\varphi^{-1}[(G, B)]\] is absolute soft group over \((X, A)\).

(v) If \(f_\varphi\) is a soft homomorphism such that \(\varphi\) is onto, then \(f_\varphi[(\ker f, A)] = (\{e_f\}, B)\).

(vi) If \((G, B)\) be a soft group over \((Y, B)\) and \(f_\varphi\) is a soft homomorphism, then 
\[f_\varphi^{-1}[(G, B)]\] is a soft group over \((X, A)\).

(vii) If \((F, A)\) be a soft group over \((X, A)\) and \(f_\varphi\) is a soft homomorphism such that \(\varphi\) is one-one, then \(f_\varphi[(F, A)]\) is a soft group over \((Y, B)\).

(viii) If \((G_1, B) \leq (Z) (G_2, B)\) and \(f_\varphi\) is a soft homomorphism, then 
\[f_\varphi^{-1}[(G_1, B)] \leq (Z) f_\varphi^{-1}[(G_2, B)]\].

(ix) If \((F_1, A) \leq (Z) (F_2, A)\) and \(f_\varphi\) is a soft homomorphism such that \(\varphi\) is one-one, then 
\[f_\varphi[(F_1, A)] \leq (Z) f_\varphi[(F_2, A)]\].

**Definition 2.14.** Let \((F, A)\) and \((G, B)\) be two soft groups over \((X, A)\) and \((Y, B)\) respectively. Then

(i) \((F, A)\) is said to be soft homomorphic to \((G, B)\), written as \((F, A) \sim (G, B)\), if for each \(\alpha \in A\) and \(\beta \in B, \exists\) a homomorphism \(f^{\alpha, \beta} : F(\alpha) \to G(\beta)\) such that 
\[f^{\alpha, \beta}[F(\alpha)] = G(\beta)\].

(ii) \((F, A)\) is said to be soft isomorphic to \((G, B)\), written as \((F, A) \simeq (G, B)\), if for each \(\alpha \in A\) and \(\beta \in B, \exists\) an isomorphism \(f^{\alpha, \beta} : F(\alpha) \to G(\beta)\) such that 
\[f^{\alpha, \beta}[F(\alpha)] = G(\beta)\].

**Definition 2.15.** (Nazmul & Samanta, 2011) Let \((N, A)\) and \((F, A)\) be two soft groups over \(X\) such that \((N, A)\) is a soft normal subgroup of \((F, A)\). Define a mapping \(\frac{F}{N} \) over \(A\) by 
\[\frac{F}{N}(\alpha) = \text{the factor group } F(\alpha) / N(\alpha), \forall \alpha \in A\]. Then the factor group \(F(\alpha) / N(\alpha)\) is a group, for each \(\alpha \in A\). So to each \(\alpha \in A\), we get a factor group \(F(\alpha) / N(\alpha)\) and thus it induces a generalized soft group which we call soft factor group and denote it by \(\left(\frac{F}{N}, A\right)\).

**Proposition 2.16.** (Nazmul & Samanta, 2011) Let \((N, A)\) be a soft normal subgroup of \((F, A)\) then for each \(\alpha \in A\) the canonical mapping \(\phi_\alpha : F(\alpha) \to F(\alpha) / N(\alpha)\), given by 
\[\phi_\alpha(\xi) = \xi N(\alpha), \xi \in F(\alpha)\], is an onto homomorphism.

**Definition 2.17.** Let \((F, A)\) and \((G, B)\) be two soft groups over \((X, A)\) and \((Y, B)\) respectively such that \((F, A)\) are soft homomorphic to \((G, B)\). Also let for each \(\alpha \in A\) and \(\beta \in B, f^{\alpha, \beta} : F(\alpha) \to G(\beta)\) be the corresponding homomorphism and \(K^{\alpha, \beta}\) be the kernel of \(f^{\alpha, \beta}\). Let for each \(\alpha \in A, K_\alpha = \bigcap_{\beta \in B} K^{\alpha, \beta}\) and define a mapping \(K : A \to P(X)\), where \(K(\alpha) = K_\alpha\).
Clearly $(K, A)$ is a soft set over $(X, A)$ and is called soft kernel corresponding to \( \{ f^{\alpha, \beta}; \alpha \in A, \beta \in B \} \).

**Remark 2.18.** Clearly $(K, A)$ is a soft normal subgroup of $(F, A)$.

**Proposition 2.19.** [Fundamental homomorphism theorem]

Let $(F, A)$ and $(G, B)$ be two soft groups over $(X, A)$ and $(Y, B)$ respectively such that $(F, A) \sim (G, B)$. Also let for each $\alpha \in A$ and $\beta \in B$, $f^{\alpha, \beta}: F(\alpha) \to G(\beta)$ be the corresponding homomorphism and $(K, A)$ be the soft kernel corresponding to \( \{ f^{\alpha, \beta}; \alpha \in A, \beta \in B \} \), then \( (F_K, A) \simeq (G, B) \).

### 3. Type-2 soft groups

In this section, we have introduced type-2 soft group and the behaviour of functional image and pre-image of type-2 soft groups under type-2 soft homomorphic mapping are discussed. Fundamental homomorphism theorem also established in this format. $X, Y$ are taken be the groups and $e_x, e_y$ are the identity elements of $X, Y$ respectively.

**Definition 3.1.** Let $[\mathcal{F}, A]$ be a type-2 soft set over $(X, A)$. Then $[\mathcal{F}, A]$ is said to be a type-2 soft group over $(X, A)$ or simply over $X$ iff $\mathcal{F}(\alpha)$ is a soft group over $X, \forall \alpha \in A$ i.e. $\mathcal{F}(\alpha) \subseteq (\hat{X}, A), \forall \alpha \in A$.

**Example 3.2.** Let $X = S_3 = \{e, (12), (13), (23), (123), (132)\}, A = \{\alpha_1, \alpha_2\}$. Now define a mapping $\mathcal{F}: A \to S_3(X, A)$ by $\mathcal{F}(\alpha_1) = \{e, (12)\}, \{e, (13)\}$ and $\mathcal{F}(\alpha_2) = \{e, (13)\}, \{e, (23)\}$. Clearly, $\mathcal{F}(\alpha)$ is a soft group over $X, \forall \alpha \in A$. Therefore, $[\mathcal{F}, A]$ is a type-2 soft group over $X$.

**Proposition 3.3.** Let $[\mathcal{F}_1, A]$ and $[\mathcal{F}_2, A]$ be two type-2 soft groups over $X$. Then their intersection $[\mathcal{F}_1, A] \cap [\mathcal{F}_2, A]$ is a type-2 soft group over $X$.

**Proof:** Since $[\mathcal{F}_1, A]$ and $[\mathcal{F}_2, A]$ be two type-2 soft groups over $X$, it follows that $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\alpha)$ are soft groups over $X, \forall \alpha \in A$.

So, $[\mathcal{F}_1 \cap \mathcal{F}_2](\alpha) = \mathcal{F}_1(\alpha) \cap \mathcal{F}_2(\alpha)$ is a soft group over $X, \forall \alpha \in A$.

Therefore, $[\mathcal{F}_1, A] \cap [\mathcal{F}_2, A]$ is a type-2 soft group over $X$.

**Proposition 3.4.** Let $[\mathcal{F}_1, A]$ and $[\mathcal{F}_2, A]$ be two type-2 soft groups over $X$. Then their AND $[\mathcal{F}_1, A] \wedge [\mathcal{F}_2, A]$ is a type-2 soft group over $X$.

**Proof:** Since $[\mathcal{F}_1, A]$ and $[\mathcal{F}_2, A]$ be two type-2 soft groups over $X$, it follows that $\mathcal{F}_1(\alpha)$ and $\mathcal{F}_2(\beta)$ are soft groups over $X, \forall \alpha, \beta \in A$.
So, \([\mathcal{F}_1 \cap \mathcal{F}_2](\alpha, \beta) = \mathcal{F}_1(\alpha) \cap \mathcal{F}_2(\beta)\) is a soft group over \(X, \forall (\alpha, \beta) \in A \times A\).

Therefore, \([\mathcal{F}_1, A] \cap [\mathcal{F}_2, A]\) is a type-2 soft group over \(X\).

**Remark 3.5.** The union of two type-2 soft groups is not a type-2 soft group in general. If \(\mathcal{F}_1(\alpha) \subseteq \mathcal{F}_2(\alpha)\) or \(\mathcal{F}_2(\alpha) \subseteq \mathcal{F}_1(\alpha), \forall \alpha \in A\), then \([\mathcal{F}_1, A] \cup [\mathcal{F}_2, A]\) is a type-2 soft group over \(X\).

**Definition 3.6.** Let \([\mathcal{F}, A], [\mathcal{F}_1, A] and [\mathcal{F}_2, A]\) be type-2 soft groups over \((X, A)\). Then

(i) \([\mathcal{F}, A]\) is said to be a type-2 identity soft group (type-2 absolute soft group) over \((X, A)\) if \(\mathcal{F}(\alpha)\) is an identity soft group (absolute soft group) over \(X, \forall \alpha \in A\) i.e. \(\mathcal{F}(\alpha) = \{(e_X, A) \mid \mathcal{F}(\alpha) = (X, A)\}, \forall \alpha \in A\).

(ii) \([\mathcal{F}, A]\) is said to be a type-2 constant soft group over \((X, A)\) if \(\exists\) a soft group \((G, A)\) over \(X\) such that \(\mathcal{F}(\alpha) = (G, A), \forall \alpha \in A\).

(iii) \([\mathcal{F}, A]\) is said to be a type-2 pseudo constant soft group over \((X, A)\) if \(\mathcal{F}(\alpha) = (X, A)\) for \(\forall \alpha \in A\).

(iv) \([\mathcal{F}, A]\) is said to be a type-2 soft subgroup (type-2 soft normal subgroup) of \([\mathcal{F}_1, A]\) if \(\mathcal{F}(\alpha) \subseteq (\tilde{e}, A)_{\tilde{\mathcal{F}}_1(\alpha), \forall \alpha \in A}\). This relation is denoted \([\mathcal{F}_1, A] \subseteq (\tilde{e}, A)_{\tilde{\mathcal{F}}_1(\alpha), \forall \alpha \in A}\).

**Proposition 3.7.** Let \([\mathcal{F}, A]\) be a type-2 soft group over \((X, A)\) and \(\{[\mathcal{H}_i, A]; i \in \Delta\}\) be a non-empty family of type-2 soft subgroups of \([\mathcal{F}, A]\), where \(\Delta\) is an index set. Then \(\cap_{i \in \Delta} [\mathcal{H}_i, A]\) is a type-2 soft subgroup of \([\mathcal{F}, A]\).

**Proof:** Since \(\mathcal{H}_i(\alpha)\) is a soft subgroup of \(\mathcal{F}(\alpha), \forall \alpha \in A\), it follows that \((\cap_{i \in \Delta} [\mathcal{H}_i, A])(\alpha) = \cap_{i \in \Delta} \mathcal{H}_i(\alpha)\) is a soft subgroup of \(\mathcal{F}(\alpha), \forall \alpha \in A\).

Hence, \(\cap_{i \in \Delta} [\mathcal{H}_i, A]\) is a type-2 soft subgroup of \([\mathcal{F}, A]\).

**Proposition 3.8.** Let \([\mathcal{F}, A]\) be a type-2 soft group over \((X, A)\) and \(\{[\mathcal{H}_i, A]; i \in \Delta\}\) be a non-empty family of type-2 soft normal subgroups of \([\mathcal{F}, A]\) where \(\Delta\) is an index set. Then \(\cap_{i \in \Delta} [\mathcal{H}_i, A]\) is a type-2 soft normal subgroup of \([\mathcal{F}, A]\).

**Proof:** Proof is similar to Proposition 3.7.

**Definition 3.9.** A soft mapping \(f_{\psi} : S_2(X, A) \rightarrow S_2(Y, B)\) is said to be type-2 soft homomorphism (type-2 soft isomorphism) if \(f_{\psi} : S_1(X, A) \rightarrow S_1(Y, B)\) is a soft homomorphism (soft isomorphism).

**Proposition 3.10.** Let \([\mathcal{F}, A], [\mathcal{F}_1, A]\) and \([\mathcal{F}_2, A]\) be type-2 soft groups over \((X, A)\) and \([G, B], [G_1, B], [G_2, B]\) be type-2 soft groups over \((Y, B)\). Also let \(f_{\psi} : S_2(X, A) \rightarrow S_2(Y, B)\) be a type-2 soft mapping.
(i) If \([\mathcal{G},B]\) be a type-2 soft group over \((Y,B)\) and \(f_{\psi}\) is a type-2 soft homomorphism, then 
\(f^{-1}_{\psi}([\mathcal{G},B])\) is a type-2 soft group over \((X,A)\).

(ii) If \([\mathcal{F},A]\) be a type-2 soft group over \((X,A)\) and \(f_{\psi}\) is a type-2 soft homomorphism such 
that \(\varphi, \psi\) are both one-one, then \(f^{-1}_{\psi}([\mathcal{F},A])\) is a type-2 soft group over \((Y,B)\).

(iii) If \([\mathcal{G},B] \subseteq [\mathcal{G},B]\) and \(f_{\psi}\) is a type-2 soft homomorphism, then 
\(f^{-1}_{\psi}([\mathcal{G},B]) \subseteq f^{-1}_{\psi}([\mathcal{G},B]).\)

(iv) If \([\mathcal{F}_1,A] \subseteq [\mathcal{F}_2,A]\) and \(f_{\psi}\) is a type-2 soft homomorphism such that \(\varphi, \psi\) are both one-one, 
then \((i)\) \(f^{-1}_{\psi}([\mathcal{F}_1,A]) \subseteq f^{-1}_{\psi}([\mathcal{F}_2,A]).\)

(v) If \([\mathcal{G},B] \subseteq [\mathcal{G},B]\) and \(f_{\psi}\) is a type-2 soft homomorphism, then 
\(f^{-1}_{\psi}([\mathcal{G},B]) \subseteq f^{-1}_{\psi}([\mathcal{G},B]).\)

(vi) If \([\mathcal{F}_1,A] \subseteq [\mathcal{F}_2,A]\) and \(f_{\psi}\) is a type-2 soft homomorphism such that \(\varphi, \psi\) are both one-one, 
then \(f_{\psi}([\mathcal{F}_1,A]) \subseteq f_{\psi}([\mathcal{F}_2,A]).\)

(vii) If \([\mathcal{F},A]\) be type-2 identity soft group over \((X,A)\) and \(f_{\psi}\) is a type-2 soft homomorphism such that \(\varphi, \psi\) are both onto, then \(f_{\psi}([\mathcal{F},A])\) is type-2 identity soft group 
over \((Y,B)\).

(viii) If \([\mathcal{G},B]\) be type-2 identity soft group over \((Y,B)\) and \(f_{\psi}\) is a type-2 soft homomorphism, then 
\(f^{-1}_{\psi}([\mathcal{G},B])\) is the type-2 constant soft group \([\mathcal{K},A]\) over \((X,A)\) where 
\(\mathcal{K}(\alpha) = (\ker(f),A).\) In particular, if \(f\) is one-one then 
\(f^{-1}_{\psi}([\mathcal{G},B])\) is type-2 identity soft group over \((X,A)\).

(ix) If \([\mathcal{F},A]\) be type-2 absolute soft group over \((X,A)\) and \(f_{\psi}\) is a type-2 soft homomorphism such that \(\varphi, \psi\) are both onto, then \(f_{\psi}([\mathcal{F},A])\) is a type-2 constant soft group over \((Y,B).\)

Also if \(f, \varphi\) and \(\psi\) are onto, then \(f_{\psi}([\mathcal{F},A])\) is type-2 absolute soft group over \((Y,B).\)

(x) If \([\mathcal{G},B]\) be type-2 absolute soft group over \((Y,B)\) and \(f_{\psi}\) is a type-2 soft homomorphism, 
then \(f^{-1}_{\psi}([\mathcal{G},B])\) is type-2 absolute soft group over \((X,A)\).

(xi) If \([\mathcal{K},A]\) be a type-2 soft set over \((X,A)\) where \(\mathcal{K}(\alpha) = (\ker(f),A)\) and \(f_{\psi}\) is a type-2 soft homomorphism such that \(\varphi, \psi\) are both onto, then \(f_{\psi}([\mathcal{K},A])\) is type-2 identity soft group over \((Y,B)\).

**Proof:**

(i) Let \([\mathcal{G},B]\) be a type-2 soft group over \((Y,B)\). Then \(\mathcal{G}(\beta)\) is a soft group over \(Y, \forall \beta \in B.\)

Let \(\alpha \in A\) and \(\psi(\alpha) = \beta.\) Now 
\(f^{-1}_{\psi}([\mathcal{G},B])(\alpha) = f^{-1}_{\psi}(\mathcal{G}(\psi(\alpha))) = f^{-1}_{\psi}([\mathcal{G}(\beta)])\) is a soft group over \(X\) by part (vi) of Proposition 2.13. So, \(f^{-1}_{\psi}([\mathcal{G},B])\) is a type-2 soft group over \((X,A)\).
(ii) Let $[F, A]$ be a type-2 soft group over $(X, A)$. Then $F(\alpha)$ is a soft group over $X, \forall \alpha \in A$. Since $\psi$ is one-one, then $\forall \beta \in B$, either $\psi^{-1}(\beta) = \phi$ or $\psi^{-1}(\beta) = \{\alpha\}$ for some $\alpha \in A$. Now for any $\beta \in B$,

$$[f_{\psi}]([F, A])(\beta) = \begin{cases} \bigcup_{\alpha \in \psi^{-1}(\beta)} \left[ f_{\psi}\left[F(\alpha)\right]\right] & \text{if } \psi^{-1}(\beta) \neq \phi \\ (\hat{\phi}, B) & \text{otherwise} \end{cases}$$

$$= \begin{cases} f_{\psi}\left[F(\alpha)\right] & \text{if } \psi^{-1}(\beta) = \{\alpha\}, \text{ since } \psi \text{ is one-one.} \\ (\hat{\phi}, B) & \text{otherwise} \end{cases}$$

is a soft group over $Y$ by part (vii) of Proposition 2.13. So, $f_{\psi}([F, A])$ is a type-2 soft group over $(Y, B)$.

(iii) Let $[G, B] \lessapprox [G_2, B]$ and $f_{\psi}$ is a type-2 soft homomorphism, then $G(\beta) \lessapprox G_2(\beta)$ and hence $f_{\psi}^{-1}[G(\beta)] \lessapprox f_{\psi}^{-1}[G_2(\beta)], \forall \beta \in B$. Also from part (i), $f_{\psi}^{-1}([G, B]), f_{\psi}^{-1}([G_2, B])$ are type-2 soft groups over $(X, A)$. Now for any $\alpha \in A$,

$$[f_{\psi}^{-1}([G, B])](\alpha) = f_{\psi}^{-1}[G(\psi(\alpha))] \lessapprox f_{\psi}^{-1}[G_2(\psi(\alpha))] = [f_{\psi}^{-1}([G, B])](\alpha).$$

Therefore,

$$f_{\psi}^{-1}([G, B]) \lessapprox f_{\psi}^{-1}([G_2, B]).$$

(iv) Let $[F_1, A] \lessapprox [F_2, A]$ and $f_{\psi}$ is a type-2 soft homomorphism such that $\varphi, \psi$ are one-one, then $F_1(\alpha) \lessapprox F_2(\alpha)$ and hence $f_{\psi}[F_1(\alpha)] \lessapprox f_{\psi}[F_2(\alpha)], \forall \alpha \in A$. Also $f_{\psi}([F_1, A]), f_{\psi}([F_2, A])$ are type-2 soft groups over $(Y, B)$. Since $\varphi, \psi$ are both one-one, then $\forall \beta \in B$, either $\psi^{-1}(\beta) = \phi$ or $\psi^{-1}(\beta) = \{\alpha\}$ for some $\alpha \in A$. Now for any $\beta \in B$,

$$[f_{\psi}]([F_1, A])(\beta) = \begin{cases} \bigcup_{\alpha \in \psi^{-1}(\beta)} \left[ f_{\psi}\left[F_1(\alpha)\right]\right] & \text{if } \psi^{-1}(\beta) \neq \phi \\ (\hat{\phi}, B) & \text{otherwise} \end{cases}$$

$$= \begin{cases} f_{\psi}\left[F_1(\alpha)\right] & \text{if } \psi^{-1}(\beta) = \{\alpha\}, \text{ since } \psi \text{ is one-one.} \\ (\hat{\phi}, B) & \text{otherwise} \end{cases}$$

$$\lessapprox \begin{cases} f_{\psi}\left[F_2(\alpha)\right] & \text{if } \psi^{-1}(\beta) \neq \phi \\ (\hat{\phi}, B) & \text{otherwise} \end{cases}$$

$$= \begin{cases} \bigcup_{\alpha \in \psi^{-1}(\beta)} \left[ f_{\psi}\left[F_2(\alpha)\right]\right] & \text{if } \psi^{-1}(\beta) = \{\alpha\}, \text{ since } \psi \text{ is one-one.} \\ (\hat{\phi}, B) & \text{otherwise} \end{cases}$$

$$= [f_{\psi}]([F_1, A])(\beta)$$

Hence $f_{\psi}([F_1, A]) \lessapprox f_{\psi}([F_2, A])$. 

\[ \text{IJRSNS is a UGC approved journal} \]
(v) Proof is similar to that of part (iii).

(vi) Proof is similar to that of part (iv).

(vii) Let $[\mathcal{F}, A]$ be type-2 identity soft group over $(X, A)$ and $f_{\psi}$ is a type-2 soft homomorphism such that $\varphi, \psi$ are onto. Then $\mathcal{F}(\alpha) = \{e_\alpha\}, A, \forall \alpha \in A$ and $\psi^{-1}(\beta) \neq \emptyset, \forall \beta \in B$. Also $f_{\psi^{-1}}([\mathcal{F}, A])$ is type-2 soft group over $(Y, B)$. Now for any $\beta \in B$, $f_{\psi^{-1}}([\mathcal{F}, A])(\beta) = \bigcup_{\alpha \in \psi^{-1}(\beta)} f_{\psi^{-1}}(\mathcal{F}(\alpha)) = \bigcup_{\alpha \in \psi^{-1}(\beta)} f_{\psi^{-1}}(\{e_\alpha\}, A) = \{e_\gamma\}, B)$. Therefore, $f_{\psi^{-1}}([\mathcal{F}, A])$ is type-2 identity soft group over $(Y, B)$.

(viii) Let $[\mathcal{G}, B]$ be a type-2 identity soft group over $(Y, B)$ and $f_{\psi}$ is a type-2 soft homomorphism. Then $\mathcal{G}(\beta) = \{e_\gamma\}, B, \forall \beta \in B$. Now for any $\alpha \in A$, $f_{\psi}^{-1}([\mathcal{G}, B])(\alpha) = f_{\psi}^{-1}(\mathcal{G}(\varphi(\alpha))) = f_{\psi}^{-1}([\{e_\gamma\}, B]) = (ker(f), A)$. Therefore, $f_{\psi}^{-1}([\mathcal{G}, B])$ is a type-2 constant soft group over $(X, A)$. Again if $f$ is one-one, then $ker(f) = \{e_\alpha\}$ and hence $f_{\psi}^{-1}([\mathcal{G}, B])(\alpha) = (ker(f), A) = \{e_\gamma\}, A, \forall \alpha \in A$. Thus, if $f$ is one-one, then $f_{\psi}^{-1}([\mathcal{G}, B])$ is type-2 identity soft group $(X, A)$.

(ix) Let $[\mathcal{F}, A]$ be type-2 absolute soft group over $(X, A)$ and $f_{\psi}$ is a type-2 soft homomorphism such that $\varphi, \psi$ are onto, then $\mathcal{F}(\alpha) = (\bar{X}, A), \forall \alpha \in A$ and $\psi^{-1}(\beta) \neq \emptyset, \forall \beta \in B$. Also from part (ii) we have, $f_{\psi^{-1}}([\mathcal{F}, A])$ is a type-2 soft group over $(Y, B)$. If $f(X) = Y_0 \subseteq Y$, then for any $\beta \in B$, we have $f_{\psi^{-1}}([\mathcal{F}, A])(\beta) = \bigcup_{\alpha \in \psi^{-1}(\beta)} f_{\psi^{-1}}([\mathcal{F}, A])(\alpha) = \bigcup_{\alpha \in \psi^{-1}(\beta)} f_{\psi^{-1}}([\bar{X}, A]) = (Y_0, B)$. Therefore, $f_{\psi^{-1}}([\mathcal{F}, A]) = [Y_0, B]$ and hence $f_{\psi^{-1}}([\mathcal{G}, B])$ is a type-2 constant soft group over $(Y, B)$. Again if $f_{\psi}$ is onto, then $f(X) = Y$ and hence $f_{\psi^{-1}}([\mathcal{F}, A]) = [\bar{Y}, B]$.

Thus, if $f_{\psi}$ is onto, then $f_{\psi^{-1}}([\mathcal{F}, A])$ is type-2 absolute soft group over $(Y, B)$.

(x) Let $[\mathcal{G}, B]$ be a type-2 absolute soft group over $(Y, B)$ and $f_{\psi}$ is a type-2 soft homomorphism. Then $\mathcal{G}(\beta) = (\bar{Y}, B), \forall \beta \in B$. Now for any $\alpha \in A$, $f_{\psi}^{-1}([\mathcal{G}, B])(\alpha) = f_{\psi}^{-1}(\mathcal{G}(\varphi(\alpha))) = f_{\psi}^{-1}([\bar{Y}, B]) = (\bar{X}, A)$. Thus, $f_{\psi}^{-1}([\mathcal{G}, B]) = [(\bar{X}, A), A]$. Therefore, $f_{\psi}^{-1}([\mathcal{G}, B])$ is the type-2 absolute soft group over $(X, A)$.

(xi) Let $[\mathcal{K}, A]$ be type-2 soft set over $(X, A)$ where $\mathcal{K}(\alpha) = (ker(f), A), \forall \alpha \in A$ and $f_{\psi}$ is a type-2 soft homomorphism such that $\varphi, \psi$ are both onto. Then $[\mathcal{K}, A]$ is a type-2 soft group over $(X, A)$ and hence $f_{\psi^{-1}}([\mathcal{K}, A])$ is a type-2 soft group over $(Y, B)$. Now for any $\beta \in B$, $\varphi^{-1}(\beta) \neq \emptyset$ (since $\varphi$ is onto) and $f_{\psi^{-1}}([\mathcal{K}, A])(\beta) = \bigcup_{\alpha \in \varphi^{-1}(\beta)} f_{\psi^{-1}}([\mathcal{K}(\alpha)]) = \bigcup_{\alpha \in \varphi^{-1}(\beta)} f_{\psi^{-1}}((ker(f), A) = \{e_\gamma\}, B)$. Therefore, $f_{\psi^{-1}}([\mathcal{K}, A]) = [\{e_\gamma\}, B)$.
Definition 3.11. Let \([\mathcal{F},A]\) and \([\mathcal{G},B]\) be two type-2 soft groups over \((X,A)\) and \((Y,B)\) respectively. Then

(i) \([\mathcal{F},A]\) is said to be soft homomorphic to \([\mathcal{G},B]\), written as \([\mathcal{F},A] \sim [\mathcal{G},B]\), if \(\mathcal{F}(\alpha) \sim \mathcal{G}(\beta)\) for each \(\alpha \in A\) and \(\beta \in B\).

(ii) \([\mathcal{F},A]\) is said to be soft isomorphic to \([\mathcal{G},B]\), written as \([\mathcal{F},A] \simeq [\mathcal{G},B]\) if \(\mathcal{F}(\alpha) \simeq \mathcal{G}(\beta)\) for each \(\alpha \in A\) and \(\beta \in B\).

Definition 3.12. Let \([\mathcal{N},A]\) and \([\mathcal{F},A]\) be two type-2 soft groups over \((X,A)\) such that \([\mathcal{N},A]\) is a type-2 normal soft subgroup of \([\mathcal{F},A]\). Define a mapping \(\mathcal{F}/\mathcal{N}\) over \(A\) by \(\mathcal{F}/\mathcal{N}(\alpha) = \) the soft factor group \(\left(\frac{\mathcal{F}_\alpha}{N_\alpha}, A\right), \alpha \in A\). Then \([\mathcal{F}/\mathcal{N},A]\) is called the type-2 soft factor group.

Proposition 3.13. Let \([\mathcal{N},A]\) be a type-2 soft normal subgroup of \([\mathcal{F},A]\) then for each \(\alpha \in A\) the canonical mappings \(\phi_\alpha^N : F_\alpha(\beta) \rightarrow F_\alpha(\beta)/N_\alpha(\beta)\) given by \(\phi_\alpha^N(\xi) = \xi N_\alpha(\beta), \xi \in F_\alpha(\beta)\), are onto homomorphisms.

Proof: Since \([\mathcal{N},A]\) is type-2 soft normal subgroup of \([\mathcal{F},A]\).

So, for each \(\alpha \in A, (N_\alpha, A)\) is soft normal subgroup of \((F_\alpha, A)\). Thus, for each \(\alpha \in A, N_\alpha(\beta)\) is normal subgroup of \(F_\alpha(\beta)\), \(\forall \beta \in A\). Therefore from Group theory, for each \(\alpha \in A\), the canonical mappings \(\phi_\alpha^N : F_\alpha(\beta) \rightarrow F_\alpha(\beta)/N_\alpha(\beta)\) are onto homomorphisms for \(\forall \beta \in A\).

Definition 3.14. Let \([\mathcal{F},A]\) and \([\mathcal{G},B]\) be two type-2 soft groups over \((X,A)\) and \((Y,B)\) respectively such that \([\mathcal{F},A] \sim [\mathcal{G},B]\). Then for each \(\alpha \in A\) and \(\beta \in B\), \((F_\alpha, A) \sim (G_\beta, B)\) and also let \((K_\alpha, A)\) be the corresponding soft kernel. Let for each \(\alpha \in A\), \((K_\alpha, A) = \cap_{\beta \in B}(K_{\alpha,\beta}, A)\) and define a mapping \(K : A \rightarrow S_1(X,A)\), where \(K(\alpha) = (K_{\alpha, A}, A), \alpha \in A\). Clearly, \([K, A]\) is a type-2 soft set over \((X,A)\) and is called type-2 soft kernel corresponding to \(\{f_{\alpha,\beta} : F_\alpha(\gamma) \rightarrow G_\beta(\delta) : \alpha, \gamma \in A; \beta, \delta \in B\}\).

Remark 3.15. Clearly \([K, A]\) is a soft normal subgroup of \([\mathcal{F}, A]\).

Proposition 3.16. (Fundamental Homomorphism Theorem)

Let \([\mathcal{F},A]\) and \([\mathcal{G},B]\) be two type-2 soft groups over \((X,A)\) and \((Y,B)\) respectively such that \([\mathcal{F},A] \sim [\mathcal{G},B]\). Also let for each \(\alpha, \gamma \in A\) and \(\beta, \delta \in B\), \(f_{\alpha,\gamma}^{\phi_\beta} : F_\alpha(\gamma) \rightarrow G_\beta(\delta)\) be the corresponding homomorphism and \([K, A]\) be the type-2 soft kernel corresponding to \(\{f_{\alpha,\beta}^{\phi_\gamma} : \alpha, \gamma \in A; \beta, \delta \in B\}\), then \([\mathcal{F}/K, A] \simeq [\mathcal{G}, B]\).

Proof: Since \([\mathcal{F},A] \sim [\mathcal{G},B]\) and hence for each \(\alpha \in A\) and \(\beta \in B\), \((F_\alpha, A) \sim (G_\beta, B)\), it follows that for each \(\gamma \in A\) and \(\delta \in B\), \(f_{\alpha,\gamma}^{\phi_\beta} : F_\alpha(\gamma) \rightarrow G_\beta(\delta)\) is an onto homomorphism. Again since
Therefore, we have from group theory, \( \exists \) an onto homomorphism \( \phi_{\alpha}^{\gamma} : F_{\alpha}(\gamma) \rightarrow F_{\alpha}(\gamma)/K_{\alpha}(\gamma) \) such that \( \phi_{\alpha}^{\gamma}(\xi) = \xi K_{\alpha}(\gamma), \forall \xi \in F_{\alpha}(\gamma) \). Let us define a mapping \( g_{\alpha,\beta}^{\gamma,\delta} : \frac{F_{\alpha}}{K_{\alpha}}(\gamma) \rightarrow G_{\beta}(\delta) \), where \( g_{\alpha,\beta}^{\gamma,\delta}[\xi K_{\alpha}(\gamma)] = f_{\alpha,\beta}^{\gamma,\delta}(\xi), \xi \in F_{\alpha}(\gamma) \).

Clearly, \( g_{\alpha,\beta}^{\gamma,\delta} \phi_{\alpha}^{\gamma} = f_{\alpha,\beta}^{\gamma,\delta} \) and \( f_{\alpha,\beta}^{\gamma,\delta} \) is an isomorphism. Thus, for each \( \alpha \in A \) and \( \beta \in B \), \( \frac{F_{\alpha}}{K_{\alpha}}(\alpha) \simeq \{G_{\beta} \} = G(\beta) \).

Therefore, \( \frac{F}{K} \simeq \{G \} \).

4. Conclusion and future work

In this paper, concepts of type-2 soft groups and type-2 soft mappings are introduced and studied some of their important properties. Constant type-2 soft mapping is defined and the behaviour of image soft sets and pre-image soft sets under this soft mapping are discussed. There is an ample scope for further research to extend it in topological group theory which have many applications in abstract integration theory viz. Haar measure, Haar integral etc. and also in manifold theory through the development of Lie groups.

References


